A COURSE
OF
PURE MATHEMATICS

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PREFACE.

THIS book has been designed primarily for the use of first year students at the Universities whose abilities reach or approach something like what is usually described as 'scholarship standard'. I hope that it may be useful to other classes of readers, but it is this class whose wants I have considered first. It is in any case a book for mathematicians: I have nowhere made any attempt to meet the needs of students of engineering or indeed any class of students whose interests are not primarily mathematical.

A considerable space is occupied with the discussion and application of the fundamental ideas of the Infinitesimal Calculus, Differential and Integral. But the general range of the book is a good deal wider than is usual in English treatises on the Calculus. There is at present hardly room for a new *Calculus* of an orthodox pattern. It is indeed not many years since there was urgent need of such a book, but the want has been met by the excellent treatises of Professors Gibson, Lamb, and Osgood, to all of which, I need hardly say, I am greatly indebted. And so I have included in this volume a good deal of matter that would find a place in any *Traité d'Analyse*, though in English books it is usually separated from the Calculus and classed as 'Higher Algebra' or 'Trigonometry'.

In the first chapter I have discussed in some detail the various classes of numbers included in the arithmetical continuum. I have not attempted to include any account of any purely arithmetical theory of irrational number, since I believe all such theories to be entirely unsuitable for elementary teaching. My aim in this chapter is a more modest one: I take the 'linear continuum' for granted and assume the existence of a definite number corresponding to each of its points; and all that I attempt to do is to analyse and distinguish the various classes of numbers whose existence these assumptions involve.

Chapters II and III probably do not present many points of novelty. The account given in Chapter II of the most important classes of functions of x is more systematic and illustrated with much greater detail than is usual in English books. I have included, mainly for the sake of completeness, a certain amount of the elements of coordinate geometry of two and three dimensions: but I have, here and throughout the book, kept geometry in a strictly subordinate position and used it merely for purposes of illustration. I have also avoided any wealth of detail in connexion with the purely formal consequences of De Moivre's Theorem, and have devoted the space thus saved to the inclusion of a good deal of matter concerning vector analysis, bilinear transformation, and so on, which seemed to me likely to be more interesting and more useful as a preparation for Chapter X.

I have endeavoured to make Chapter IV one of the principal features of the book. The notion of a limit is one that has always presented grave difficulties to mathematical students even of great ability. It has been my good fortune during the last eight or nine years to have a share in the teaching of a good many of the ablest candidates for the Mathematical Tripos; and it is very rarely indeed that I have encountered a pupil who could face the simplest problem involving the ideas of infinity, limit, or continuity, with a vestige of the confidence with which he would deal with questions of a different character and of far greater intrinsic difficulty. I have indeed in an examination asked a dozen candidates, including several future Senior Wranglers, to sum the series $1 + x + x^2 + \dots$, and not received a single answer that was not practically worthless—and this from men quite capable of solving difficult problems connected with the curvature and torsion of twisted curves.

I cannot believe that this is due solely to the nature of the subject. There are difficulties in these ideas, no doubt: but they are not so great as many other difficulties inherent in mathematics that every young mathematician completely overcomes. The fault is not that of the subject or of the student, but of the text-book and the teacher. It is not enough for the latter, if he wishes to drive sound ideas on these points well into the mind of his pupils, to be careful and exact himself. He must be prepared not merely to tell the truth, but to tell it elaborately and ostentatiously. He must drill his pupils in 'infinity' and 'continuity', with an abundance of written exercises and examples, as he drills them at present in poles and polars or symmetric functions or the consequences of De Moivre's theorem. Then and only then he may hope that accurate thought in connexion with these matters will become an integral part of their ordinary mathematical habit of mind. It is this conviction that has led me to devote so much space to the most elementary ideas of all connected with limits, to be purposely diffuse about fundamental points, to illustrate them by so elaborate a system of examples, and to write a chapter of fifty pages without advancing beyond the ordinary geometrical series.

It is not necessary for me to say much about the general plan of the next four chapters. The two chapters on the Calculus are no doubt more difficult than the rest of the book. I have perhaps been inconsistent in the standards that I have adopted: but I have been influenced by the feeling that I shall have few readers who will not already have acquired some familiarity with the technique of the Calculus from other sources. I felt this particularly when I was writing the sections on integration. I also felt that the student is apt to carry away from the books in general use the quite mistaken impression that all methods of integration are essentially of a tentative and haphazard character. I have therefore deliberately given an account of the theory more systematic and general than would be suitable for a normal first course in the Calculus.

Chapters IX and X are devoted to the theory of the logarithm and exponential, starting from the definition of the logarithm as an integral. It was the desire to write an elementary account of this theory that originally led me to begin the book, and I have generally decided my choice of what was to be included in the earlier chapters by a consideration of what theorems would be wanted in the last two.

I regard the book as being really elementary. There are plenty of hard examples (mainly at the ends of the chapters): to these I have added, wherever space permitted, an outline of the solution. But I have done my best to avoid the inclusion of anything that involves really difficult ideas. For instance, I make no use of the 'principle of convergence': uniform convergence, double series, infinite products, are never alluded to: and I prove no general theorems whatever concerning the inversion of limit-operations—I never even define $\frac{d^2f}{dx\,dy}$ and $\frac{d^2f}{dy\,dx}$. In the last two chapters I have occasion once or twice to integrate a power-series, but I have confined myself to the very simplest cases and given a special discussion in each instance. Anyone who has read this book will be in a position to read with profit Mr Bromwich's *Infinite Series*, where a full and adequate discussion of all these points will be found.

It will be found that certain classes of theorems and examples that are prominent in many English books are here conspicuous by their absence. I may refer particularly to the standard theorems concerning the expression of the trigonometrical functions as infinite products or series of partial fractions, and to that familiar type of example the gist of which lies in the 'picking out of coefficients' from some combination of infinite series. The proofs of these results depend upon general theorems that seemed to me intrinsically too difficult to be included in a book professing to be at the same time rigorous and elementary: and I am on the whole of opinion that, if any proposition is too difficult to be proved properly, its statement and application had better be postponed. I am well aware that there is much to be said on the opposite side. A very plausible case can be made out for the habitual exercise of the student in the application of results whose proof is too difficult for his full comprehension. But I have found that I cannot myself write a book on those lines: nor am I fully convinced that such exercise is either necessary or desirable. After all there are plenty of theorems which are *not* too difficult to prove: and, if anyone believes that a sufficient variety of analytical training cannot be based upon them, I hope that my collections of Miscellaneous Examples may do something to convince him. I may say that it is only in these collections that examples of the character of 'problems' will be found. The sets of examples inside each chapter consist either of perfectly straightforward applications of the preceding 'book-work', or of summaries of parts of the theory for which there was no room in the text. They include many important theorems, some indeed to which reference is frequently made later in the book. No one can be more convinced than I am of the value of 'examples' designed merely to train the student's memory and powers of manipulation: but I see no reason why all examples should necessarily be trivial. I trust, however, that readers will not find it irritating to be referred back from the middle of a section in large type to an example in an earlier chapter. My decision as to whether a result should appear in the text or in the examples has always been based upon the relation that it bears to the general theorems in connexion with which it is first proved rather than upon the amount of use that is made of it later on.

I have throughout laid particular stress upon points that do not seem to me to be emphasized sufficiently in the text-books in general use, and passed rapidly over others that are of equal importance but stand in no such danger of neglect. Here again I have been influenced by the consideration that this book is likely to be used in conjunction with others rather than as a first text-book in any particular subject.

There are two respects in which I have diverged from the usually accepted notation and that seem of sufficient importance to be noticed here. I have entirely rejected the index notation for inverse functions ($\cos^{-1} x$, $\cosh^{-1} x$) in favour of the usual Continental notation (arc cos x, arg cosh x or arg ch x). And I have followed Mr Leathem and Mr Bromwich in always writing

$$\lim_{n \to \infty}, \quad \lim_{x \to \infty}, \quad \lim_{x \to a}$$

and not $\lim_{n=\infty}$, $\lim_{x=\infty}$, $\lim_{x=a}$. This last change seems to me one of considerable importance, especially when '∞' is the 'limiting value'. I believe that to write '$n = \infty$, $x = \infty$' (as if anything ever were 'equal to infinity'), however convenient it may be at a later stage, is in the early stages of mathematical training to go out of one's way to encourage incoherence and confusion of thought concerning the fundamental ideas of analysis.

The word 'quantity' occurs occasionally in the earlier chapters. It should be in each case altered to 'number'. Unfortunately I arrived at the decision never to use the term 'quantity' only after the earlier sheets had been passed for press.

The books to which I am most indebted (besides the treatises on the Calculus already mentioned) are Mr Bromwich's *Infinite*

Series and M. J. Tannery's *Leçons d'Algèbre et d'Analyse.* I must also acknowledge my obligations to a number of friends who have been kind enough to assist me in the preparation of the book. Mr Bromwich has read the whole of it (except Chapter III) either in manuscript or in proof, and a good deal of it twice; and I am indebted to him for corrections and suggestions on almost every page. Mr Berry read Chapters I, II, III, IX and X in manuscript, Professor J. E. Wright Chapters I, II, and III, and Dr Whitehead Chapters I and IV, and all gave me much valuable advice. In particular the earlier part of Chapter IV has been practically rewritten in consequence of Dr Whitehead's suggestions. I have also changed a good deal of Chapter VI in consequence of suggestions received from Dr Askwith. My thanks are also due to Messrs H. W. Turnbull and E. H. Neville, of Trinity College, who have between them read all the proofs and verified the examples: to the latter I am additionally indebted for the figures that appear in the Miscellaneous Examples to Chapter X. Finally I must express my gratitude to the readers and officials of the University Press for their close attention and unfailing courtesy.

G. H. HARDY.

TRINITY COLLEGE, CAMBRIDGE,
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CHAPTER I.

REAL VARIABLES.

1. The aggregate of rational numbers and their representation on a straight line. On a straight line L, produced indefinitely in both directions, we take a segment A_0A_1 of any length. We call A_0 *the origin*, or *the point* 0, and A_1 *the point* 1.

We now mark off a series of points

$$\ldots A_{-m-1},\ A_{-m},\ \ldots,\ A_{-1},\ A_0,\ A_1,\ \ldots,\ A_n,\ \ldots$$

along L, so that

$$\ldots = A_{-m-1}A_{-m} = \ldots = A_{-1}A_0 = A_0A_1 = \ldots,$$

each segment being measured from left to right along L.

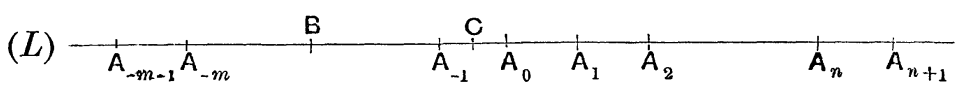

FIG. 1.

Then $$\frac{A_0A_n}{A_0A_1} = n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

if n is any positive integer.

We will now agree that *length* is to be regarded as *a magnitude capable of sign*, positive if the length is measured in one direction along L (e.g. from B to C) and negative if measured in the other (from C to B), so that $CB = -BC$. We take the positive direction for the measurement of length to be from left to right.

Then $$\frac{A_0A_{-n}}{A_0A_1} = -\frac{A_{-n}A_0}{A_0A_1} = -n.$$

Hence the equation (1) is true for all integral values of n, positive or negative.

For the sake of uniformity we adopt the convention that (1) is also true when $n = 0$, in which case it reads

$$\frac{A_0A_0}{A_0A_1} = 0.$$

That is to say, we agree to regard BB, which is not, properly speaking, a segment at all, as a segment of no length.

Now let us take any positive proper fraction in its lowest terms, for example p/q, where p and q are positive integers without any common factor, and $p < q$. We divide A_0A_1 into q equal parts by points of division which it is natural to denote by

$$A_0,\ A_{1/q},\ A_{2/q},\ \ldots,\ A_{p/q},\ \ldots,\ A_{(q-1)/q},\ A_1.$$

It is evident that

$$\frac{A_0A_{p/q}}{A_0A_1} = \frac{p}{q} \qquad \text{(2)}.$$

We thus obtain points on the line L corresponding to all such proper fractions p/q.

Any improper fraction may be expressed in the form $n + (p/q)$, where n is a positive integer and p/q a proper fraction. If we take a point $A_{n+(p/q)}$ such that $A_nA_{n+(p/q)} = A_0A_{p/q}$, it is evident that $\dfrac{A_0A_{n+(p/q)}}{A_0A_1} = n + \dfrac{p}{q}$: and if we thus find points $A_{n+(p/q)}$ corresponding to all possible positive values of n, p, and q, we shall have a point A_f corresponding to all possible positive integral or fractional values of f, and such that

$$\frac{A_0A_f}{A_0A_1} = f \qquad \text{(3)}.$$

Finally, if $-f$ is a negative fraction, proper or improper, we take A_{-f} so that $A_{-f}A_0 = A_0A_f$, or

$$\frac{A_0A_{-f}}{A_0A_1} = -\frac{A_0A_f}{A_0A_1} = -f.$$

Thus we are able to determine a point A_r corresponding to *any integral or fractional value of* r, *positive or negative*, and such that

$$\frac{A_0A_r}{A_0A_1} = r \qquad \text{(4)}.$$

If we take, as is natural, the length A_0A_1 as our unit of length, so that $A_0A_1 = 1$, the equation (4) becomes

$$A_0A_r = r \quad \text{..........................(5)}.$$

DEFINITIONS. *Any fraction $r = p/q$, where p and q are positive or negative integers, is called a* **rational number.**

The points A_r of the line L, which correspond to the rational numbers r in the manner explained above, are called the **rational points** *of the line.*

We can suppose (i) that p and q have no common factor, as if they have a common factor we can divide each of them by it, and (ii) that q is positive, since

$$p/(-q) = (-p)/q, \quad (-p)/(-q) = p/q.$$

The notion of a rational number obviously includes as a particular case that of an integer, since any integer may be expressed as a fraction whose denominator is unity.

Examples I. 1. If r and s are rational numbers, $r+s$, $r-s$, rs, and r/s are rational numbers, unless in the last case $s=0$ (when r/s is of course meaningless).

2. If P and Q are rational points, and PQ is divided into any number of equal parts, each of the points of division is a rational point.

3. If λ, m, and n are positive rational numbers, $\lambda(m^2 \sim n^2)$, $2\lambda mn$, and $\lambda(m^2+n^2)$ are positive rational numbers. Hence show how to determine any number of right-angled triangles the lengths of all of whose sides are rational.

4. Any terminated decimal represents a rational number whose denominator contains no factors other than 2 or 5. Conversely, any such rational number can be expressed, and in one way only, as a terminated decimal.

[The general theory of decimals will be considered in Chap. IV.]

5. The positive rational numbers may be arranged in the form of a simple series as follows:

$$\tfrac{1}{1}, \tfrac{2}{1}, \tfrac{1}{2}, \tfrac{3}{1}, \tfrac{2}{2}, \tfrac{1}{3}, \tfrac{4}{1}, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}, \dots$$

Show that p/q is the $[\frac{1}{2}(p+q-1)(p+q-2)+q]$th term of the series.

[In this series every rational number is repeated indefinitely. Thus 1 occurs as $\frac{1}{1}$, $\frac{2}{2}$, $\frac{3}{3}$, ... We can of course avoid this by omitting every number which has already occurred in a simpler form, but then the problem of determining the precise position of p/q becomes more complicated.]